PERIOD–VI

M12CH21

TOPIC

21

Exponential and Logarithmic Functions

21.1. EVALUATING EXPONENTIAL FUNCTIONS

Most of the functions (polynomial, rational, radical, etc.) we have studied thus far have been algebraic functions. Algebraic functions involve basic operations, powers, and roots. In this chapter, we discuss exponential functions and logarithmic functions. The following table illustrates the difference between algebraic functions and exponential functions:

Function	***Variable is in the***	***Constant is in the***	***Example***	***Example***
Algebraic	Base	Exponent	$f(x) - x^2$	$g(x) - x^{1/3}$
Exponential	Exponent	Base	$F(x) - 2^x$	$G(x) = \left(\frac{1}{3}\right)^x$

Definition: Exponential function

An exponential function with base b is denoted by

$$f(x) = b^x$$

where b and x are any real numbers such that $b > 0$ and $b \neq 1$.

Note:

- We eliminate $b = 1$ as a value for the base because it merely yields the constant function $f(x) = 1^x = 1$.
- We eliminate negative values for b because they would give non-real-number values such as $(-9)^{1/2} = \sqrt{-9} = 3i$.
- We eliminate $b = 0$ because 0^x corresponds to an undefined value when x is negative.

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Sometimes the value of an exponential function for a specific argument can be found by inspection as an exact number.

x	-3	-1	0	1	3
$F(x) = 2^x$	$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$	$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$	$2^0 = 1$	$2^1 = 2$	$2^3 = 1$

Example 1. *Let* $f(x) = 3^x$, $g(x) = \left(\frac{1}{4}\right)^x$ *and* $h(x) = 10x^2$. *Find the following values:*

(*a*) $f(2)$ (*b*) $f(\pi)$ (*c*) $g\left(-\frac{3}{2}\right)$

(*d*) $h(2.3)$ (*e*) $f(0)$ (*f*) $g(0)$

If an approximation is required, approximate to four decimal places.

Solution.

(*a*) $f(2) = 3^2 - \mathbf{9}$ (*b*) $f(\pi) = 3^\pi \approx \mathbf{31.5443}$*

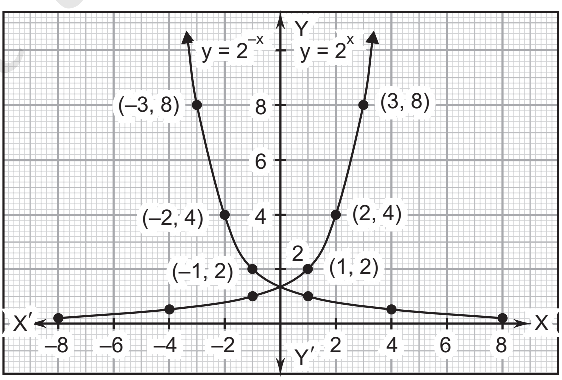
(*c*) $f\left(-\frac{3}{2}\right) = \left(\frac{1}{4}\right)^{-3/2} = 4^{3/2} = (\sqrt{4})^3 = 2^3 = \mathbf{8}$

(*d*) $h(2, 3) = 10^{2.3-2} = 10^{0.3} \approx \mathbf{1.9953}$

(*e*) $f(0) = 3^0 = \mathbf{1}$ (*f*) $g(0) = \left(\frac{1}{4}\right)^0 = \mathbf{1}$

21.2. GRAPHS OF EXPONENTIAL FUNCTIONS

Let's graph two exponential functions, $y = 2^x$ and, $y = 2^{-x} = \left(\frac{1}{2}\right)^2$ by plotting points.



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x	$y = 2^x$	(x, y)
2	$2^{-3} = \frac{1}{2^2} = \frac{1}{4}$	$\left(-2, \frac{1}{4}\right)$
– 1	$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$	$\left(-1, \frac{1}{2}\right)$
0	$2^0 = 0$	(0, 1)
1	$2^1 = 2$	(1, 2)
2	$2^2 = 4$	(2, 4)
3	$2^3 = 8$	(3, 8)

x	$y = 2^{-x}$	(x, y)
– 3	$2^{-(-3)} = 2^3 = 8$	(– 3, 8)
– 2	$2^{-(-2)} = 2^2 = 4$	(– 2, 4)
– 1	$2^{-(-1)} = 2^1 = 2$	(– 1, 2)
0	$2^0 = 1$	(0, 1)
1	$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$	$\left(1, \frac{1}{2}\right)$
2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$	$\left(2, \frac{1}{4}\right)$

Notice that both graphs' y-intercept is (0, 1) (as shown to the left) and neither graph has an x-intercept. The x-axis is a horizontal asymptote for both graphs. The following box summarizes general characteristics of the graphs of exponential functions.

21.3. PROCEDURE FOR GRAPHING $f(x) = b^x$

Step 1: Label the point (0, 1) corresponding to the y-intercept $f(0)$.

Step 2: Find and label two additional points corresponding to $f(-1)$ and $f(1)$.

Step 3: Connect the three points with a smooth curve with the x-axis as the horizontal asymptote.

Example 2. *Graphing Exponential Functions for $b > 1$.*

Graph the function $f(x) = 5^x$.

$f(0) = 5^x = 1$

$f(0) = 5^1 = 1$

$f(-1) = 5^{-1} = \frac{1}{5} = 0.2$

Solution. Step 1: Label the y-intercept (0, 1)

Step 2: Label the point (1, 5).

Label the point (–1, 0.2).

Step 3: Sketch a smooth curve through the three points with the x-axis as a horizontal asymptote.

Domain: $(-\infty, \infty)$

Range: $(0, \infty)$

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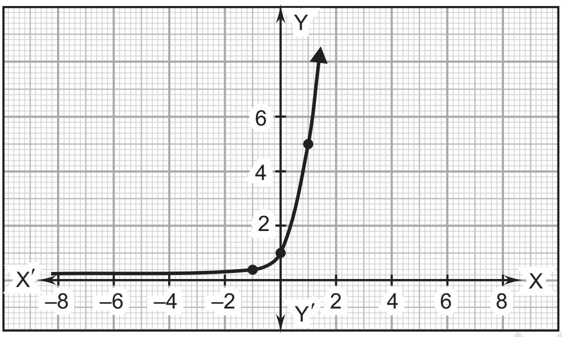

21.4. THE NATURAL BASE e

Any positive real number can serve as the base for an exponential function. A particular irrational number, denoted by the letter *e*, appears as the base in many applications, as you will soon see when we discuss continuous compounded interest. Although you will see 2 and 10 as common bases, the base that appears most often is e, because e, as you will come to see in your further studies of mathematics, is the natural base. The exponential function with base *e*, $f(x) = e^x$, is called the exponential function or the natural exponential function. Mathematicians did not pull this irrational number out of a hat. The number *e* has many remarkable properties, but most simply, it comes from evaluating the expression $\left(1+\frac{1}{m}\right)^m$ as m gets large (increases without bound).

	m	$\left(1+\frac{1}{m}\right)^m$
	1	2
	10	2.59374
$e \approx 2.71828$	100	2.70481
	1000	2.71692
	10000	2.71815
	100000	2.71827
	1000000	2.71828

Calculators have an e^x button for approximating the natural exponential function.

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Example 3. *Evaluating the Natural Exponential Function*

Evaluate $f(x) = e^x$ for the given x-values. Round your answers to four decimal places.

(*a*) $x = 1$ (*b*) $x = –$

(*c*) $x = 1.2$ (*d*) $x = –0.47$

Solution.

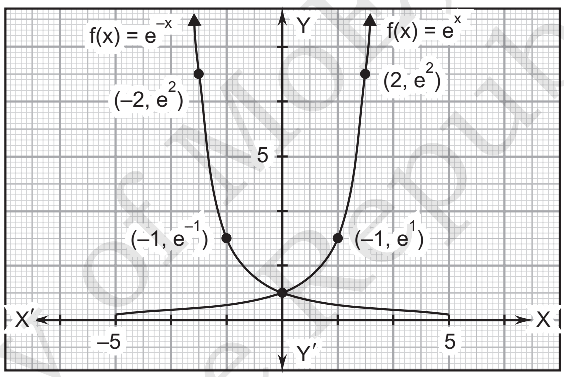
(*a*) $f(1) = e^1 \approx 2.718281828 \approx$ **2.7183**

(*b*) $f(-1) = e^1 \approx 0.367879441 \approx$ **0.3679**

(*c*) $f(1.2) = e^{1.2} \approx 3.320116923 \approx$ **3.3201**

(*d*) $f(-0.47) = e^{0.47} \approx 0.625002268 \approx$ **0.6250**

Like all exponential functions of the form $f(x) = b^x$, $f(x) = e^x$ and $f(x) = e^{-x}$ have (0, 1) as their y-intercept and the x-axis as a horizontal asymptote as shown in the figure on the right.



21.5. APPLICATIONS OF EXPONENTIAL FUNCTIONS

Exponential functions describe either growth or decay. Populations and investments are often modeled with exponential growth functions, while the declining value of a used car and the radioactive decay of isotopes are often modeled with exponential decay functions. In Section, various exponential models will be discussed. In this section, we discuss doubling time, half-life, and compound interest.

21.6. DOUBLING TIME GROWTH MODEL

The doubling time growth model is given by

$$P = P_0 2^{t/d}$$

where P = Population at time t, P_0 = Population at time $t = 0$

d = Doubling time

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Note that when $t = d$, $P = 2P_0$ (population is equal to twice the original).

The units for P and P_0 are the same and can be any quantity (people, dollars, etc.). The units for t and d must be the same (years, weeks, days, hours, seconds, etc.).

In the investment scenario with Maria and David, P_0 = \$5,000 and d = 10 years, so the model used to predict how much money the original \$5,000 investment yielded is $P = 5000(2)^{t/10}$. Maria retired 40 years after the original investment, t = 40, and David retired 30 years after the original investment, t = 30.

Maria: $P = 5000(2)^{40/10} = 5000(2)^4 = 5000(16) = 80{,}000$

David: $P = 5000(2)^{30/10} = 5000(2)^3 = 5000(8) = 40{,}000$

Example 4. *In 2004 the population in Kazakhstan, a country in Asia, reached 15 million. It is estimated that the population doubles in 30 years. If the population continues to grow at the same rate, what will the population be in 2024? Round to the nearest million.*

Solution. Write the doubling model. $P = P_0 2^{t/d}$

Substitute P_0 = 15 million, d = 30 years, and t = 20 years.

$P = 15(2)^{20/30}$

Simplify, $P = 15(2)^{2/3} \approx 23.8110$

In 2024, there will be approximately 24 million people in Kazakhstan.

Example 5. *The radioactive isotope of potassium 42K, which is used in the diagnosis of brain tumors, has a half-life of 12.36 hours. If 500 milligrams of potassium 42 are taken, how many milligrams will remain after 24 hours? Round to the nearest milligram.*

Solution. Write the half-life formula. $A = A_0\left(\frac{1}{2}\right)^{t/h}$

Substitute A_0 = 500 mg, h = 12.36 hours, t = 24 hours.

$$A = 500\left(\frac{1}{2}\right)^{24/12.36}$$

Simplify $A \approx 500(0.2603) \approx 130.15$

After 24 hours, there are approximately 130 milligrams of potassium 42 left.

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21.7. SIMPLE INTEREST

Simple interest was defined where the interest I is calculated based on the principal P, the annual interest rate r, and the time t in years, using the formula I = Prt.

If the interest earned in a period is then reinvested at the same rate, future interest is earned on both the principal and the reinvested interest during the next period. Interest paid on both the principal and interest is called compound interest.

21.8. COMPOUNT INTEREST

If a principal P is invested at an annual rate *r* compounded *n* times a year, then the amount A in the account at the end of t years is given by

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

The annual interest rate *r* is expressed as a decimal.

The following list shows the typical number of times interest is compounded:

Annually $n = 1$ Monthly $n = 12$

Semiannually $n = 2$ Weekly $n = 52$

Quarterly $n = 4$ Daily $n = 365$

Example 6. *If $3,000 is deposited in an account paying 3% compounded quarterly, how much will you have in the account in 7 years?*

Solution. Write the compound interest formula.

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

Substitute P = 3000, $r = 0.03$, $n = 4$, and $t = 7$.

$$A = 3000\left(1+\frac{0.03}{4}\right)^{(4)(7)}$$

Simplify

You will have $3,698.14 in the account.

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21.9. THE NUMBER e

The number e is defined as the value that $(1 + 1/n)^n$ approaches as n becomes large. (In calculus this idea is made more precie through the concept of a limit). The table shows the values of the expression $(1 + 1/n)^n$ for increasingly large values of n.

m	$\left(1+\frac{1}{n}\right)^n$
1	2.00000
5	2.48832
10	2.59374
100	2.70481
1000	2.71692
10000	2.71815
100000	2.71827
1000000	2.71828

It appears that rounded to five decimal places, $e = 2.71828$; in fact, the approximate value to 20 decimal places is

$$e \approx 2.71828182845904523536$$

It can be shown that e is an irrational number, so we cannot write its exact value in decimal form.

21.10. THE NATURAL EXPONENTIAL FUNCTION

The number e is the base for the natural exponential function. Why use such a strange base for an exponential function? It might seem at first that a base such as 10 is easier to work with. We will see, however that in certain applications the number e is the best possible base. In this section we study how e occurs in the description of compound interest.

The natural exponential function is the exponential finction

$$f(x) = e^{x}$$

with base e. It is often referred to as the exponential function.

Since $2 < e < 3$, the graph of the natural exponential function lies between the graphs of $y = 2^x$ and $y = 3^x$ as shown in Figure.

Scientific calculations have a special key for the function $f(x) = e^x$. We use this key in the next example.

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Example 7. *Evaluate each expression rounded to five decimal places*

(*a*) e^3 (*b*) $2e^{-0.53}$ (*c*) $e^{4.8}$

Solution. We use e^x key on a calculator to evaluate the exponential function.

(*a*) $e^3 \approx 20.08554$ (*b*) $2e^{-0.53} \approx 1.17721$

(*c*) $e^{4.8} \approx 121.51042$

21.11. LOGARITHMIC FUNCTIONS

Recall from section that a function *f* has an inverse function if and only if *f* is one-to-one. In the preceding section, you learned that $f(x) = a^x$ is one-to-base, so *f* must have an inverse function. This inverse function is the logarithmic function with base *a* and is denoted by the symbol $\log_a x$ read as "log base *a* of *x*".

Defintion of Logarithmic Function with Base a

The equations $y = \log_a x$ and $x = a^y$ are equivalent, where the first equation is written in logarithmic form and the second equation is written in exponential form. For example, $2 = \log_3 9$ is equivalent to $9 = 3^2$ and $5^3 = 12$ is equivalent to $\log_5 125 = 3$.

When evaluating logarithms, remember that a *logarithm is an exponent*. This means that $\log_a x$ is the exponent to which *a* must be raised to obtain *x*. For example, $\log_2 8 = 3$ because 2 raised to the third power is 8.

Example 8. *Evaluate each logarithm at the given value of x.*

(*a*) $f(x) = \log_2 x, x = 32$ (*b*) $f(x) = \log_3 x, x = 1$

(*c*) $f(x) = \log_4 x, x = 2$ (*d*) $f(x) = \log_{10} x, x = \frac{1}{100}$

Solution. (*a*) $f(32) = \log_2 32 = 5$ because $2^5 = 32$

(*b*) $f(1) = \log_3 1 = 0$ because $3^0 = 1$

(*c*) $f(2) = \log_4 2 = \frac{1}{2}$ because $4^{1/2} = \sqrt{4} = 22$

(*d*) $f\left(\frac{1}{100}\right) = \log_{10}\frac{1}{100} = -2$ because $10^{-2} = \frac{1}{10^2} = \frac{1}{100}$

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21.12. PROPERTIES OF LOGARITHMS

1. $\log_a 1 = 0$ because $a^0 = 1$
2. $\log_a a = 1$ because $a^1 = a$
3. $\log_a a^x = x$ and $a^{\log_a x} = x$ Inverse properties
4. If $\log_a x = \log_a y$, then $x = y$ One-to-One Property

Example 9. *Using properties of Logarithms*

(*a*) *To solve* $\log_3 x = \log_3 12$ *for x use the One-to-One property.*

(*b*) *To solve* $\log (2x + 1) = \log 3x$ *for x use the One-to-One property.*

Solution. (*a*)

$\log_3 x = 12$ Write original equation

$x = 12$ One to One property

(*b*)

$\log (2x + 1) = \log 3x$ Write original equation

$2x + 1 = 3x$ One to One property

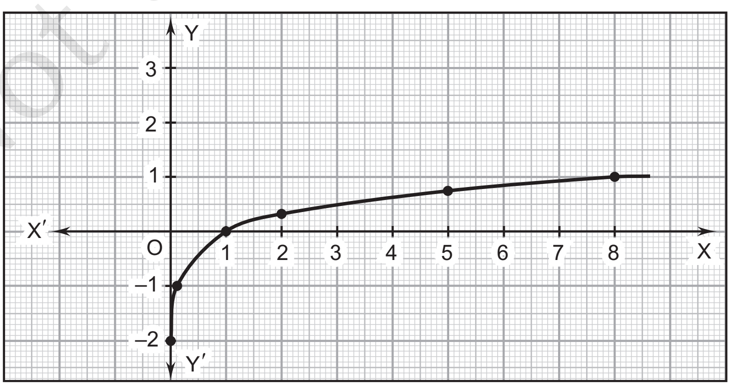
$1 = x$ Subtract $2x$ from each

Example 10. *Sketch the graph of f(x) = log x. Identify the vertical asymptote.*

Solution. Being by constructing a table of values. Note that some of the values can be obtained without calculator by using the properties of logarithms. Other required a calculator.

	Without calculator				***With calculator***		
x	$\frac{1}{100}$	$\frac{1}{10}$	1	10	2	5	8
$f(x) = \log x$	– 2	– 1	0	1	0.301	0.699	0.903

Next, plot the points and connect them with a smooth curve, as shown in figure below. The vertical asymptote is $x = 0$ (y-axis).



21.13. LOGARITHM BASE PROPERTIES

Before we proceed ahead for logarithm properties, we need to revise the law of exponents, so that we can compare the properties.

For exponents, the laws are

- Product rule: $a^m.a^n = a^{m+n}$
- Quotient rule: $a^m/a^n = a^{m-n}$
- Power of a power: $(a^m)^n = a^{mn}$

Now let us learn the properties of logarithm functions.

Product Property

If a, m and n are positive integers and $a \neq 1$, then

$\log_a(mn) = \log_a m + \log_a n$

Thus, the log of two number m and n, with base 'a' is equal to the sum of log m and log n with the same base.

Example 11. log_3 (9.25)

Solution. $\log_3 (9) + \log_3 (27)$

$= \log_3 (3^2) + \log_3 (3^3)$

$= 2 + 3$(By property: $\log_b bx = x$)

$= 5$

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21.14. QUOTIENT PROPERTY

If m, n and a are positive integers and $a \neq 1$, then

$\log_a(m/n) = \log_a m - \log_a n$

In the above expression, the logarithm of a quotient of two positive numbers m and n results in a difference of log of m and log n with the same base 'a'.

Example 12. log_2 $(21/8)$.

Solution. $\log_2 (21/8) = \log_2 21 - \log_2 8$

21.15. POWER RULE

If a and m are positive numbers, $a \neq 1$ and n is a real number, then

$$\log_a m^n = n \log_a m$$

The above property defines that logarithm of a positive number m to the power n is equal to the product of n and log of m.

21.16. CHANGE OF BASE RULE

If m, n and p are positive numbers, $n \neq 1$, $p \neq 1$, then:

$$\log_n m = \log_p m / \log_p n$$

Example 13. $\log_2 10 = \log_p 10/\log_p 2$

Solution. Reciprocal rule

If m and n are the positive numbers other than 1, then;

$$\log_n m = 1/\log_m n$$

21.17. LOGARITHMIC FUNCTIONS AND ITS TYPES

The logarithm is defined as a quantity that represents the powe in which the base (fixed number) is raised to produce a given number. The general representation of the logarithmic function is:

In general, the two different types of logarithmic functions are

- Common Logarithmic Function
- Natural Logarithmic Function

In common logarithmic function, the base of the logarithmic function is 10. Log_{10} or log represents this function.

In natural logarithmic function, the base of the logarithmic function is e. Log_e or ln represents this function.

21.18. VALUE OF LOG 1 TO 10 FOR LOG BASE 10

The value of log 1 to 10 (common logarithm $\log_{10} x$)

Common Logarithm to a Number ($\log_{10} x$)	***Log Value***
Log 1	0
Log 2	0.3010
Log 3	0.4771
Log 4	0.6020
Log 5	0.6989
Log 6	0.7781
Log 7	0.8450
Log 8	0.9030
Log 9	0.9542

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Example 14. *Find the value of* $\log_{10}(75/16) - 2\log 10(5/9) + \log(32/243)$.

Solution. We know that $\log_a m^n = n \log_a m$

$$\log_a (p/q) = \log_a p - \log_a q$$

$$\log_a (mn) = \log_a m + \log_a n$$

Using these logarithmic rules, we have

$$\log_{10}(75/16) - 2\log_{10}(5/9) + \log(32/243)$$
$$= \log_{10}(75/16) - \log_{10}(5/9)^2 + \log(32/243)$$
$$= \log_{10}(75/16) - \log_{10}(25/81) + \log(32/243)$$
$$= \log_{10}[(75/16) \times (32/243)] - \log_{10}(25/81)$$
$$= \log_{10}(50/81) - \log_{10}(25/81)$$
$$= \log_{10}[(50/81)/(25/81)]$$
$$= \log_{10}[(50/81) \times (25/81)]$$
$$= \log_{10}[(50/81) \times (81/25)]$$
$$= \log_{10} 2 = 0.3010$$

$$\therefore \quad \log_{10}(75/16) - 2\log_{10}(5/9) + \log(32/243) = 0.3010$$

Example 15. *Evaluate* $\log_{10}(124416)$.

Solution. $\log_{10}(124416) = \log_{10}(512 \times 243)$
$$= \log_{10}(29 \times 35)$$
$$= \log_{10} 29 + \log_{10} 35$$
$$= 9\log_{10} 2 + 5\log_{10} 3$$
$$= 9 \times 0.3010 + 5 \times 0.4771$$
$$= 2.709 + 2.3855$$
$$= 5.0945$$

$$\therefore \quad \log_{10}(122416) = 5.0945$$

21.19. CHANGE OF BASE OF FORMULA

The change of base formula helps to rewrite the logarithm in terms of another base log. Change of base formula is used in the evaluation of log and have another base 10.

$$\log_b x = \frac{\log_d x}{\log_d x}$$

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Example 16. *Simplify* log_{32} *16.*

Solution. Given $\log_{32} 16$

Using change of base formula,

$$\log_{32} 16 = \frac{\log_{10} 16}{\log_{10} 32} = \frac{\log_{10} 2^4}{\log_{10} 2^5} = \frac{4 \log_{10} 2}{5 \log_{10} 2} = \frac{4}{5}$$

21.20. LOGARITHMS TO BASES OTHER THAN 10

Let $b > 0$ and $b \neq 1$. Then x is the logarithm of a to the base b written $x = \log_b a$, if and only if $b^x = a$.

For example, because $3^5 = 243$, you can write $5 = \log_3 243$. This is read "5 is the logarithm of 243 with base 3" or "5 is log 243 to the base 3" or "5 is the log base 3 of 243". At the right are some other power of 3 and the related logs of the base of 3

Exponential Form	***Logarithmic Form***
$3^4 = 81$	$\log_3 81 = 4$
$3^3 = 27$	$\log_3 27 = 3$
$3^2 = 9$	$\log_3 9 = 2$
$3^1 = 3$	$\log_3 3 = 1$
$3^{0.5} = 81$	$\log_3 \sqrt{3} = 0.5$
$3^0 = 1$	$\log_3 1 = 0$
$3^{-1} = 81$	$\log_3 \left(\frac{1}{3}\right) = -1$
$3^{-2} = 81$	$\log_3 \left(\frac{1}{9}\right) = 4$
$3^y = 81$	$\log_3 x = y$

21.21. SOLVING LOGARITHMIC EQUATIONS

To solve a logarithmic equation, it often helps to use the definition of logarithm to rewrite the equation in exponential form

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Example 17. *Solve for h :log$_4$ = $\frac{3}{2}$.*

Solution. $4^{\frac{3}{2}} = h$ definition of logarithm

$8 = h$ Rational Exponent Theorem

EXERCISE

1. Graphing exponential functions for $b < 1$. Graph the function $f(x) = \left(\frac{2}{5}\right)^x$
2. Graphing exponential functions with base e. Graph the function $f(x) = 3 + e^x$.
3. Graphing the exponential functions. Sketch the graph of each function. State the domain, range and asymptote.

   (*a*) $f(x) = e^{-x}$ (*b*) $g(x) = 3e^{0.5x}$
4. Use a calculator to evaluate the function $f(x) = \log x$ at each value of x.

   (*a*) $x = 10$ (*b*) $x = \frac{1}{3}$

   (*c*) $x = -2$

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